Introduction to Replica Theory

6.1 Replica Solution of REM

Take $E_j \sim \mathcal{N}(0, N/2)$

$$Z = \sum_{j=1}^{2^N} e^{-\beta E_j}$$

1) $$Z^n = \sum_{i_1 \cdots i_n}^{2^N} e^{-\beta E_{i_1} \cdots -\beta E_{i_n}} = \sum_{i_1 \cdots i_n}^{2^n} e^{-\beta \sum_{ja} \delta_{j i_a} E_j}$$

2) $$\langle Z^n \rangle_{E_j} = \sum_{i_1 \cdots i_n}^{2^N} \mathbb{E}_{E_j}\left[e^{-\beta \sum_j E_j \delta_{j i_a}} \right] = \sum_{i_1 \cdots i_n} \prod_j \mathbb{E}_{E_j}\left[e^{-\beta E_j \sum_a \delta_{j i_a}} \right]$$

$$= \sum_{i_1 \cdots i_n} \prod_j \exp\left[\frac{N\beta^2}{4} \sum_{a,b} \delta_{j i_a} \delta_{j i_b} \right]$$

$$= \sum_{i_1 \cdots i_n} \exp\left[\frac{N\beta^2}{4} \sum_{a,b} \delta_{i_a i_b} \right]$$

$$-\frac{E_j^2}{N} - \beta E_j \delta_{j i_a}$$

$$= -\frac{1}{N}\left(E_j + \frac{\beta}{2} N \delta_{j i_a}\right)^2$$

$$= 0 + \frac{N\beta^2}{4} \sum_{i_a, i_b} \delta_{j i_a} \delta_{j i_b}$$

Z for a new replicated system

i) No longer disordered

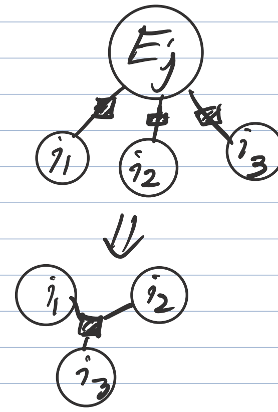
ii) E is β-dep

iii) Replicas interact. Lowest energy when $i_1 = \cdots = i_n$

$$E = -\frac{N\beta}{4} n^2$$

The elements $i_1 \ldots i_n$ are independent conditional on the sample

Upon marginalizing, they become dependent



Given a configuration $i_1 \cdots i_n$ of the replicas, the energy depends only on the matrix

$$Q_{ab} := \delta_{i_a i_b}$$

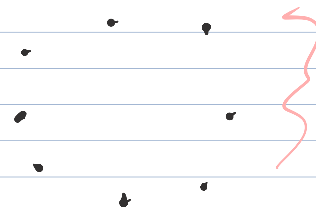
$$\Rightarrow \quad \mathbb{E}\, Z^n = \sum_Q N_N(Q) \exp\left[\frac{\beta^2 N}{4} \sum_{a,b} Q_{ab}\right]$$

$2^{n(n-1)}$ symmetric $\{0,1\}$ matrices w/ ones on the diagonal

# of configs $(i_1 \cdots i_n)$ w/ overlap matrix Q (large)

total # configs $= 2^{nN}$

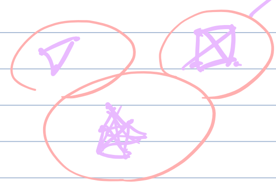
$\Rightarrow N_N(Q) \sim 2^{2N/(n-1)}$



But this is treating replicas as undirected graphs

Partition

Really, they are g clusters of fully connected graphs



$\Rightarrow Q_{ab}$ is 1 if a, b in same group, 0 else

$\Rightarrow \binom{n}{n_1 \cdots n_G}$ such choices of Q

For each such Q, $2^N(2^N-1)\cdots(2^N-G+1)$ choices of i_a for the groups

$\sim 2^{Ng}$ for $N \gg n \geq G$

$\Rightarrow \mathbb{E}[Z^n] = \sum_Q \exp[N g(Q)]$

$g(Q) = \frac{\beta^2}{4} \sum_{ab} Q_{ab} + G \log 2$

$g(Q)$ is symmetric under permutation

$Q_{ab} \to Q^{\pi}_{ab} = Q_{\pi(a)\pi(b)}$

replica symmetry

Dominant saddle is one where $Q^{\pi} = Q \; \forall \pi$
ie $Q = 1 1^T$ or $Q = \mathbb{1}$

1) $Q_{RS,0} = \mathbb{1}$

indep replicas

$N(Q_{RS,0}) = 2^N \cdots (2^N - n + 1)$

$\Rightarrow s(Q_{RS,0}) = n \log 2$

$\Rightarrow g(Q_{RS,0}) = n\left(\frac{\beta^2}{4} + \log 2\right)$

2) $Q_{RS,1} = 1 1^T \Rightarrow s(Q_{RS,1}) = \log 2$

locked replicas

$\Rightarrow g(Q_{RS,1}) = \frac{n^2 \beta^2}{4} + \log 2$

For $n > 1$: $\beta > \beta_c \Rightarrow Q_{RS,1}$ wins (locked)
$\beta < \beta_c \Rightarrow Q_{RS,0}$ wins (indep)

$\beta_c(n) = \sqrt{\frac{4 \log 2}{n}}$

Vice versa for $n < 1$

RS ansatz:

$\mathbb{E}[Z^n] = \exp[N \max(g_0, g_1)]$

For $n < 1$, this result is phisically strange

g_1 does not go to zero so we'd get: $\mathbb{E}\, Z^0 \neq 1$

Replica method: use the min for $n < 1$!

Example:

$$Z_{toy}(n) = \left(\frac{2\pi}{N}\right)^{n(n-1)/4}$$

$$Z_{toy} = \int \prod_{a\neq b} dQ_{ab} \exp\Big[\underbrace{-\frac{N}{2}\sum_{a\neq b} Q_{ab}^2}_{N g(Q)}\Big]$$

assume RS:

$$Q^*_{a\neq b} = q_0 \Rightarrow g(Q^*) = -\frac{1}{2} q_0^2\, n(n-1)$$

$$q_0 \Rightarrow Z_{toy} \doteq 1 \qquad \text{(correct)}$$

For $n<1$ this is a min, not a max!

So lets take $Q_{RS,1}$ for $\beta > \beta_c$

$Q_{RS,0}$ for $\beta < \beta_c$

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z = \lim_{N\to\infty} \lim_{n\to 0} \frac{1}{N}\frac{1}{n} \log \mathbb{E}[Z^n]$$

as $n \to 0$
$\beta_c \to \infty$
so $Q_{RS,0}$ wins

$$= \lim_{n\to 0} \frac{1}{n} g_0(n,\beta) = \frac{\beta^2}{4} + \log 2$$

One-step RSB:

Groups of size x, $x \mid n$

$$Q_{ab} = \begin{cases} 1 & \text{if } a,b \text{ in the same group} \\ 0 & \text{otherwise} \end{cases}$$

$$(i_1 \ldots i_x)\ (i_{x+1} \ldots i_{2x}) \ldots (i_{n-x+1} \ldots i_n)$$

$$2^N \qquad (2^N - 1) \qquad \ldots \qquad (2^N - \tfrac{n}{x} + 1)$$

$$\Rightarrow s = \frac{n}{x} \log 2$$

$$n^2 \cdot \frac{x}{n}$$

$$\Rightarrow g_{RSB} = \frac{\beta^2}{4} n x + \frac{n}{x} \log 2$$

$$\frac{\partial g}{\partial x} = \frac{\beta^2}{4} n - \frac{1}{x^2}\log 2 \Rightarrow x^* = \frac{2\sqrt{\log 2}}{\beta} = \frac{\beta_c}{\beta}$$

$$\Rightarrow g^*_{RSB} = \beta\sqrt{\log 2}\, n$$

$$\Rightarrow \mathbb{E}\log Z = \beta \log 2$$

correct for $\beta > \beta_c$

when we take $\beta < \beta_c$ use $x = 1$

we'll show that as $n \to 0$ $1 \le x \le n$ becomes $0 \le x \le 1$

Another view of the replica method:

Recall $\frac{1}{N}\log \underset{\xi}{\mathbb{E}}\left[e^{Nt\xi(x)}\right] =: \psi_N(t)$ "Moment generating fn for ξ"

$$\lim_{N\to\infty} \psi_N = \sup_{\xi\in\mathbb{R}} t\xi - I(\xi)$$

Now take $\xi = \frac{1}{N}\log Z$

$$\Rightarrow \psi(n) = \lim_{N\to\infty}\frac{1}{N}\log \mathbb{E} Z^n$$

$$\Rightarrow \mathbb{E}[Z^n] = \int d\xi\, \exp[-N I(\xi) - N\beta n \xi] \doteq \exp[-N \inf(I(\xi) + \beta n \xi)]$$

Calculating Y

$$Y = \sum_{j=1}^{2^N} \mu_\beta(j)^2 = \underset{E_j}{\mathbb{E}}\left[Z^{-2}\sum_{j=1}^{2^N} e^{-2\beta E_j}\right]$$

$$= \lim_{n\to 0}\mathbb{E}\left[Z^{n-2}\sum_{j=1}^{2^N} e^{-2\beta E_j}\right]$$

$$= \lim_{n\to 0} \mathbb{E}\left[\sum_{i_1 \cdots i_{n-2}} e^{-\beta E_{i_1} \cdots - \beta E_{i_{n-2}}} \sum_{j=1}^{2^N} e^{-2\beta E_j} \right]$$

$$= \lim_{n\to 0} \mathbb{E}\left[\sum_{i_1 \cdots i_n} e^{-\beta E_{i_1} \cdots - \beta E_{i_n}} \delta_{i_{n-1} i_n} \right]$$

symmetrize

$$= \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \mathbb{E}\left[\sum_{i_1 \cdots i_n} e^{-\beta E_1 \cdots E_{i_n}} \delta_{i_a i_b} \right]$$

$$= \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \langle Q_{ab} \rangle$$

as before

$$= \lim_{n\to 0} \frac{1}{n(n-1)} \underbrace{\sum_{a\neq b} \left[Q_{1RSB} \right]_{a,b}} = \lim_{n\to 0} \frac{n(x^*-1)}{n(n-1)} = 1 - x^* = 1 - \frac{\beta_c}{\beta}$$

$$x^* = \frac{\beta_c}{\beta}$$

8.2 p-spin glass model

$$H = -\sum_{i_1 < \cdots < i_p} J_{i_1 \cdots i_p} \sigma_{i_1} \cdots \sigma_{i_p} \qquad i \in \{1, \ldots, 2^N\}$$

$$\Rightarrow \mathbb{E}_J Z^n = \sum_{\{\sigma_i^a\}} \prod_{i_1 < \cdots < i_p} \mathbb{E} \exp\left[\beta J_{i_1 \cdots i_p} \sum_{a=1}^{n} \sigma_{i_1}^a \cdots \sigma_{i_p}^a \right]$$

lemma: $\langle \exp \lambda X \rangle_{X \sim N(0,\Delta)} = \exp \frac{\Delta \lambda^2}{2}$

$$= \sum_{\{\sigma_i^a\}} \prod_{i_1 < \cdots < i_p} \exp\left[\frac{\beta^2}{4} \frac{p!}{N^{p-1}} \sum_{a,b} \sigma_{i_1}^a \sigma_{i_1}^b \cdots \sigma_{i_p}^a \sigma_{i_p}^b \right]$$

$$\doteq \sum_{\{\sigma_i^a\}} \exp\left[\frac{\beta^2}{4}\frac{1}{N^{p-1}}\sum_{a,b}\left(\sum_i \sigma_i^a \sigma_i^b\right)^p\right]$$

Want $\quad Q_{ab} = \frac{1}{N}\sum_i \sigma_i^a \sigma_i^b$

$$\rightarrow \sum_{Q_{ab}} N(Q) \exp\left[\frac{\beta^2}{4}N \cdot \sum_{a,b} Q_{a,b}^p\right]$$

$$= \sum_{Q_{ab}} N(Q) \exp\left[N\frac{\beta^2 n}{4} + \frac{N\beta^2}{2}\sum_{a<b} Q_{ab}^p\right]$$

before it was $\delta_{a,b}$

ab repeated $\Rightarrow \sum_{a<b}$

Take $\quad 1 = \int \delta\left(Q_{ab} - \frac{1}{N}\sum_{i=1}^{N}\sigma_i^a\sigma_i^b\right) dQ_{ab}$

$$= N\int dQ_{ab}\frac{d\lambda_{ab}}{2\pi} \exp\left[-i\lambda_{ab}\left(NQ_{ab} - \sum_i \sigma_i^a \sigma_i^b\right)\right]$$

$$\Rightarrow \mathbb{E}Z^n \doteq \int \prod_{a<b} dQ_{ab}\, d\lambda_{ab} \exp\left[-iN\lambda_{ab}Q_{ab} + \frac{N\beta^2 n}{4} + \frac{\beta^2 N}{2} Q_{ab}^p\right]$$

$$\sum_{\{\sigma_i^a\}} \exp\left[i\lambda_{ab}\,\sigma_i^a\sigma_i^b\right]$$

$$\doteq \int \prod_{a<b} dQ_{ab}\, d\lambda_{ab} \exp\left[-NG(Q,\lambda)\right]$$

$$G = w\cdot Q - \frac{\beta^2 n}{4} - \frac{\beta^2}{2}Q^p - \log\sum_{\{\sigma_i^a\}} e^{w_{ab}\sigma_i^a\sigma_i^b}$$

$w := i\lambda$

$\frac{\delta}{\delta Q} \Rightarrow w_{ab} = p\frac{\beta^2}{2}Q_{ab}^{p-1}$

$$\frac{\delta}{\delta \omega} \Rightarrow Q_{ab} = \frac{\sum_{\{\sigma_i^a\}} \sigma_i^a \sigma_i^b \, e^{\sum_{a'<b'} \omega_{a'b'} \sigma_i^{a'} \sigma_i^{b'}}}{\sum_{\{\sigma_i^a\}} e^{\omega_{a'b'} \sigma_i^{a'} \sigma_i^{b'}}} =: \langle Q_{ab} \rangle_n$$

$$RS: \quad Q = q \Rightarrow \omega = p\frac{\beta^2}{2} q^{p-1},$$

$$q = \mathbb{E}_z \tanh^2(z\sqrt{\omega})$$

$$\Rightarrow \underbrace{q = \mathbb{E}_z \tanh^2 z\beta \sqrt{\frac{p q^{p-1}}{2}}}_{\text{self-consistent}}$$